Turbulent 4-wave Interaction of Two Type of Waves

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We consider turbulent 4-wave interaction of two types of waves: acoustic waves (dispersion $\omega = k$) and electromagnetic-type waves (dispersion $\Omega^2 = m^2 + p^2$). For large wave vectors ($k \gg m$), when the dispersion of EM-type waves becomes quasiacoustic, the stationary spectra are obtained following a standard Zakharov approach. In the small wave number region ($k, p \ll m$) we derive nonlinear differential Kompaneets-type kinetic equation for arbitrary distributions of the interacting fields and find when this equation has Kolmogorov-type power law solutions.

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I. INTRODUCTION

In this work we consider 4-wave interaction of two types of waves: one with acoustic dispersion $\omega = k$, called massless field, and another with electromagnetic-type dispersion $\Omega^2 = m^2 + p^2$, called massive field (ω, Ω are frequencies, k, p - wave vectors of the waves, the phase speed of the acoustic waves was set to unity). This is a fairly common case of wave interaction. It includes, for example, interaction of electrons with photons and electromagnetic fields with Langmuir waves. We assume that the interacting fields are scalar fields obeying quantum Bose-Einstein statistics. Then the collisional integral becomes a nonlinear functional in occupation numbers of both types of waves. We are interested in finding stationary solutions of such collision integral. In particular we will be looking for power law type solutions ($n \propto k^\alpha$, $N \propto p^\beta$) which can exist in the inertial ranges k, $p \gg m$ or k, $p \ll m$.

A standard procedure to find power law solutions of the kinetic equation is through Zakharov conformal transformations [1]. This method provides a simple rule for finding stationary power law solutions in the inertial region for differential scattering (when the change in energy or momentum in each scattering is small). For coupled kinetic equations the Zakharov transformation in the general cannot be done. Tractable interactions include (i) 3-wave interaction of waves with similar dispersion relations (using conformal transformations), (ii) 3-wave interaction of high frequency and low frequencies waves [1] (using a diffusion approximation).

In this work we consider coupled kinetic equations for 4-wave interaction of waves. In the "ultrarelativistic" regime, when the wave vectors of both particles is much larger than the mass $k\,,p \gg m$, both types of waves have similar acoustic-type dispersion and thus can be viewed as selfinteraction between quasiacoustic waves. The stationary spectra then can be found using conformal transformations. In the oposite limit of small wave vectors, k, $p \ll m$, different dispersion laws of interacting waves break the homogeneity of the kernel of the kinetic equation and do not allow exact Kolmogorov-type solutions. In this case we expand the collision integral in small changes of energy in each collision. Then the collision integral may be simply written as a nonlinear diffusion equation. This method was first applied by Kompaneets [6] who derived a nonlinear equation for the interaction of the low frequency boson particles (photons) with nonrelativistic classical electrons which were assumed to be in thermodynamic equilibrium due to quick relaxation by the long range Coulomb forces.

The "ultrarelativistic" regime, when both interacting waves have quasiacoustic dispersion requires a careful consideration - for exactly acoustic dispersion the weak turbulence theory may not be applicable (e.g., [2]- [4]). Balk [2] and Newell and Aucoin [3] argued that in the multidimensional case the angular dispersion can justify the applicability of the weak turbulence theory to acoustic waves, while Zakharov and Sagdeev [4] resorted to small corrections to the acoustic dispersion. We follow the latter approach: in the limit $p \gg m$ the correction to the dispersion of EM-type waves will provide the dispersion required for the

final time of interaction of acoustic waves and will ensure that kinetic approach is applicable.

The primary purpose of this work is to derive a low frequency diffusion approximation for the 4-wave interaction for arbitrary distribution functions of both interacting waves and to find Kolmogorov-type solutions in the inertial ranges. In case of quantum fields, the Kolmogorov-type power law solution may be possible only in the inertial ranges, where energies of the interacting particles are not close to the only quantity with dimension of energy - m and the occupation numbers satisfy $n\,, N \gg 1$ or $n\,, N \ll 1$.

We start with a Hamiltonian of the system of two interacting waves in terms of wave amplitudes a_k and A_p

$$\begin{aligned}\mathcal{H} = &\int \omega_k a_k a_k^* d^3k + \int \Omega_p A_p A_p^* d^3p + \\ &\frac{1}{4}\int T_{kpk'p'} a_k^* A_p^* a_{k'} A_{p'} \delta(\mathbf{k}+\mathbf{p}-\mathbf{k}'-\mathbf{p}') d^3k d^3p d^3k' d^3p'\end{aligned} \tag{1}$$

We assume the simplest form of the relativistically invariant matrix element (e.g., [7])

$$T_{kpk'p'} = \frac{g^2}{\omega_k \Omega_p \omega_{k'} \Omega_{p'}} \tag{2}$$

Then the kinetic equations for occupation numbers n_k, N_p are

$$\begin{pmatrix} \frac{\partial n_k}{\partial t} \\ \frac{\partial N_p}{\partial t} \end{pmatrix} = \frac{\pi}{2}\int T^2_{kpk'p'} F[n,N] \delta(\omega_k + \Omega_p - \omega_{k'} - \Omega_{p'}) \delta(\mathbf{k}+\mathbf{p}-\mathbf{k}'-\mathbf{p}') \begin{pmatrix} d^3p \\ d^3k \end{pmatrix} d^3k' d^3p' \tag{3}$$

where

$$F[n,N] = N_{p'} n_{k'} \left(n_k + N_p + 1\right) - n_k N_p \left(N_{p'} + n_{k'}\right) \tag{4}$$

II. APPLICABILITY OF THE KINETIC EQUATION

For the applicability of the weak turbulence theory (based on random phase approximation) we need that the typical interaction time $t_{int} \approx 1/(TN_{tot})$, (where $N_{tot} = \int d^3p N_p$ and T is a matrix element for amplitudes) be much larger than chaotization time for the

phases of waves $t_{collision} = 1/\Delta k v_g$ (v_g is the group velocity and the wave packet is centered on $k_0 \pm \Delta k$) and diffusion time $t_{diff} = 1/\Delta k^2 v'_g$.

For a power law $\omega \sim k^\alpha$ we have $v'_g \sim \alpha(\alpha-1)k^{\alpha-2}$, so $t_{collision} \sim \frac{k}{\Delta k}\frac{1}{\alpha\omega}$ which is can be very short - of the order $1/\omega$. On the other hand the diffusion time $t_{diff} = (\Delta k/k)^{-2}/(\alpha(\alpha-1)\omega)$. Which obviously unapplicable to acoustic turbulence. Still, if $\alpha \neq 1$, then the diffusion time t_{diff} also can be very short. This case (nonacoustic turbulence) is applicable for small wave vectors: $k \ll m$.

$$\frac{t_{diff}}{t_int} = TN/kc \sim \frac{g^2 nk}{m} \tag{5}$$

where we estimated $T(k \ll m) = g^2/(km)$, $N = nk^3$. Thus the applicability of the kinetic equation in this case requires

$$n \ll \frac{m}{g^2 k} \tag{6}$$

which for $k \leq m$ reducesto $n \ll \frac{1}{g^2}$.

On the other hand for semiacoustic waves with $k \gg m$ we find $\omega = k\,c(1+\frac{m^2}{k^2})$ we have $\omega'' = \frac{m^2}{k^3}$, $t_{diff} = \left(\frac{k}{\Delta k}\right)^2 \frac{k}{m^2} \approx \frac{k}{m^2}$, then the applicability criterion is

$$\frac{t_{diff}}{t_{int}} = \frac{TNk}{m^2} = \frac{g^2 nk^2}{m^2} \ll 1 \tag{7}$$

(the matrix element in this limit is $T = g^2/k^2$). This will eventually break at

$$k_{max} = \frac{m}{g\sqrt{n}} \tag{8}$$

Thus for applicability of a kinetic equation in a range $m \ll k \ll k_{max}$ the same condition should be satisfied, $n \ll \frac{1}{g^2}$.

We can also estimate when the higher terms will be unimportant in the kinetic equation. The ratio of the second order terms of expansion in g^2 to the first order is typically

$$[2]/[1] = g^4 k^3 n^2/\omega_1\omega_2\omega \tag{9}$$

For acoustic turbulence $k \gg m$ the frequencies are $\omega_i \sim k$, this gives

$$[2]/[1]_a = g^4 n^2 \ll 1 \text{ if } n \ll \tfrac{1}{g^2} \tag{10}$$

while for small wave vectors $\omega_1 = m\ \omega_2 = k$

$$[2]/[1] = g^4 n^2 k/m \ll 1 \text{ if } n \ll \tfrac{1}{g^2} \tag{11}$$

Thus we conclude that for existence of inertial range where weak turbulence theory is applicable the occupation numbers should satisfy

$$N_p, n_k \ll \frac{1}{g^2} \tag{12}$$

III. INTERACTION OF PARTICLES AT LARGE WAVE VECTORS

In the limit $k, p \gg m$ ("relativistic particles") the dispersion law for the EM-type waves become almost acoustic

$$\Omega \approx k\left(1 + \frac{m^2}{2k^2}\right) \tag{13}$$

The applicability of the weak turbulence theory to the interaction of acoustic wave in more than one dimension is a long standing question. In 1-D weak turbulence theory is not applicable since the interaction time between two waves is infinite, so that mutual influence of waves becomes large. In larger dimensions there were claims ([3], [2]) that angular dispersion may limit the interaction time sufficiently to allow weak turbulence approximation. Another approach is to allow for small corrections to the dispersion which will limit the interaction time of wave packets and allows us to implement methods of the weak turbulence theory regardless of angular dispersion [4].

Assuming the weak turbulence theory is applicable (Eq. (12)) we can find stationary solutions to the kinetic equation using Zakharov conformal transformations. In the limit $k, p \gg m$ the dispersion laws $\omega \propto k^\alpha$ and $\Omega \propto p^\alpha$ with $\alpha = 1$, and the interaction coefficient in this limit is a homogeneous function of the order -2:

$$T_{kpk'p'}(\lambda\mathbf{k}, \lambda\mathbf{p}, \lambda\mathbf{k}', \lambda\mathbf{p}') = \lambda^p T_{k123}(\mathbf{k}, \mathbf{p}, \mathbf{k}', \mathbf{p}'), \quad p = -2 \tag{14}$$

Zakharov transformations then gives stationary solutions

$$n_P,\, N_P \propto \omega^{-5/3}, \quad n_Q,\, N_Q \propto \omega^{-4/3} \tag{15}$$

Here P and Q symbolize Kolmogorov spectra with constant flux of energy and constant flux of action. It is possible to check that collision integral converges for both spectra, so that the interaction is local.

The sign of fluxes sign$[P] > 0$ and sign$[Q] < 0$ imply that two possible stationary spectra include the constant energy flux flowing to large k or constant flux of particles to small k. Realization of a particular spectra (Eqns (15) depends on the boundary condition at the limits of the inertial range, e.i., by the location of the sinks and sources of energy and action. If the source of energy and of action are not at $k = 0$ and $k = \infty$ correspondingly, then, depending on values of p and α, the integrals of energy or action converge or diverge. Using criteria for the stability of Kolmogorov spectra [8], we conclude that for sources located at $k_0, p_0 \gg m$ it is *Kolmogorov-in-action* case that is realized (the action integral diverges at large k). In this case the turbulence is Kolmogorov in action - particles are flowing to small k (and may eventually be stored in Bose condensate). For $k, p < k_0, p_0$ a stationary Kolmogorov-in-action spectrum will form, while for for $k, p > k_0, p_0$ there will be a selfsimilar "thermal wave" type solution carrying energy to large k and p:

$$n_k = \frac{1}{t^{\frac{3}{7}}} n_0(\frac{\omega}{t^{\frac{5}{7}}}), \quad \text{for } k > k_0 \tag{16}$$

where it is eventually dissipated [8]. Thus, in the "thermal wave" the mean frequency increases with time $\propto t^{\frac{5}{7}}$.

This situation (flow of particles to small wave numbers, where stationary spectrum is formed, and flow of energy to large wave numbers) is somewhat similar to interaction of Langmuir waves where in the low frequency region the dominant nonlinear process for Langmiur waves is the exchange of virtual ion sound waves which leads accumulation of particles with small k - Langmiur collapse.

IV. INTERACTION OF PARTICLES AT SMALL WAVE VECTORS

A. Kompaneets approximation

In the limit $k, p \ll m$ the change in energy in each collision is small, so we can expand the collision integral in small changes of the energies of particles:

$$\begin{aligned} \omega_{k'} &= \omega_k + \Delta \\ \Omega_{p'} &= \Omega_p - \Delta \end{aligned} \tag{17}$$

where Δ to the first order in ω/m is given by

$$\Delta = -\frac{\omega\ (\omega\ (1 - \mathbf{n}\cdot\mathbf{n}') + \mathbf{p}\cdot(\mathbf{n}-\mathbf{n}'))}{m + \omega\ (1 - \mathbf{n}\cdot\mathbf{n}') - \mathbf{p}\cdot\mathbf{n}'} \tag{18}$$

Expansion of $F[n, N]$ to the second order in Δ gives

$$\begin{aligned} &F[n,N] = \Delta\left(N(N+1)\frac{\partial n}{\partial\omega} - n(n+1)\frac{\partial N}{\partial\omega}\right) + \\ &\frac{\Delta^2}{2}\left(N(N+1)\frac{\partial^2 n}{\partial\omega^2} + n(n+1)\frac{\partial^2 N}{\partial\omega^2} - 2\frac{\partial N}{\partial\omega}\frac{\partial n}{\partial\omega}\,(n+N+1)\right) \end{aligned} \tag{19}$$

In the following we concentrate on the kinetic equation for particles n. In the kinetic equations we next separate angular and momentum integrations

$$\begin{aligned} &I_{coll}(k) = 4\pi\sigma_0 \int_0^\infty p^2 dp\left[\left\{\int \frac{d\mathbf{\Omega_p}}{4\pi}\frac{d\mathbf{\Omega_{n'}}}{4\pi}\Delta((k,\mathbf{p},\mathbf{n}')^2\right\}\left(\frac{\partial N}{\partial\Omega}n(n+1) - \frac{\partial n}{\partial\omega}N(N+1)\right)\right. \\ &\left.+\frac{1}{2}\left\{\int \frac{d\mathbf{\Omega_p}}{4\pi}\frac{d\mathbf{\Omega_{n'}}}{4\pi}\Delta((k,\mathbf{p},\mathbf{n}')^2\right\}\left(n(n+1)\frac{\partial^2 N}{\partial\Omega^2} + N(N+1)\frac{\partial^2 n}{\partial^2\omega} - 2\frac{\partial N}{\partial\Omega}\frac{\partial n}{\partial\omega}\,(N+n+1)\right)\right] \end{aligned} \tag{20}$$

where

$$\sigma_0 = \frac{1}{16\pi}\frac{|M_{fi}|^2}{m^2} \tag{21}$$

Denoting the angle averages

$$\begin{aligned} I_1(p,k) &= \sigma_0 \int \frac{d\mathbf{\Omega_p}}{4\pi}\frac{d\mathbf{\Omega_{n'}}}{4\pi}\Delta((k,\mathbf{p},\mathbf{n}') = \frac{k\ (4\,k^2 - 3\,k\,m + p^2)}{3\,m^2}\sigma_0 \\ I_2(p,k) &= \sigma_0 \int \frac{d\mathbf{\Omega_p}}{4\pi}\frac{d\mathbf{\Omega_{n'}}}{4\pi}\Delta((k,\mathbf{p},\mathbf{n}')^2 = \frac{2\,k^2\ (2\,k^2 + p^2)}{3\,m^2}\sigma_0 \end{aligned} \tag{22}$$

We can rewrite collision integral in the form

$$
\begin{aligned}
I_{coll}(k) = 4\pi\sigma_0 \int_0^\infty p^2 dp \Bigg[I_1(p,k) \left(\frac{\partial N}{\partial \Omega} n(n+1) - \frac{\partial n}{\partial \omega} N(N+1) \right) \\
+\frac{1}{2} I_2(p,k) \left(n(n+1) \frac{\partial^2 N}{\partial \Omega^2} + N(N+1) \frac{\partial^2 n}{\partial^2 \omega} - 2 \frac{\partial N}{\partial \Omega} \frac{\partial n}{\partial \omega} (N+n+1) \right) \Bigg]
\end{aligned} \tag{23}
$$

There is a also a constraint condition on the equation (23) - it sould conserve total number of particles, which is an integral of motion of the Hamiltonian (Eq. 1):

$$
\int d^3k I_{\rm coll}(k) = 0 \tag{24}
$$

Partially integrating and collecting terms with different powers of distribution function we find

$$
\begin{aligned}
&\int d\omega d\Omega N n \left[\left(\frac{\partial}{\partial \Omega} - \frac{\partial}{\partial \omega} \right) i_2 + \frac{1}{2} \left(\frac{\partial^2}{\partial^2 \Omega} + \frac{\partial^2}{\partial^2 \omega} - 2 \frac{\partial^2}{\partial \Omega \partial \omega} \right) i_2 \right] + \\
&\int d\omega d\Omega N^2 n \frac{\partial}{\partial \omega} \left[-i_1 + \frac{1}{2} \left(\frac{\partial}{\partial \omega} - \frac{\partial}{\partial \Omega} \right) i_2 \right] + \\
&\int d\omega d\Omega n^2 N \frac{\partial}{\partial \Omega} \left[i_1 - \frac{1}{2} \left(\frac{\partial}{\partial \omega} - \frac{\partial}{\partial \Omega} \right) i_2 \right] = 0
\end{aligned} \tag{25}
$$

where

$$
i_1 = k^2 \frac{\partial k}{\partial \omega} p^2 \frac{\partial p}{\partial \Omega} I_1, \quad i_2 = k^2 \frac{\partial k}{\partial \omega} p^2 \frac{\partial p}{\partial \Omega} I_2 \tag{26}
$$

Each term in Eq. (25) should be equal to zero. This gives a relation between the coefficients i_1 and i_2:

$$
i_1 = \frac{1}{2} \left(\frac{\partial i_2}{\partial \Omega} - \frac{\partial i_2}{\partial \omega} \right) \tag{27}
$$

This is an important relation that is one of the main results of the work. It relates the coefficients in the diffusion expansion of the collisional integral independently of the particular form of the interaction.

Using relation (27) we can simplify considerably the collision integral (Eq. 3). Partially integrating we find that collision integrals take a particularly simple diffusion type form:

$$I_{\rm coll}(k) = 2\pi\sigma_0 \int p^2 dp \frac{\partial}{\partial\omega}\left(\omega^2 I_2 \left[n' N(N+1) - N' n(n+1)\right]\right)$$
$$I_{\rm coll}(p) = 2\pi\sigma_0 \int k^2 dk \frac{\partial}{\partial\Omega}\left(\sqrt{\Omega} I_2 \left[n' N(N+1) - N' n(n+1)\right]\right) \quad (28)$$

For massless particles n and massive particles N correspondingly.

Equations (28) are the diffusion-type equations for the low frequency interaction of two types of quantum waves - "massless" acoustic waves "n" and "massive" EM waves N. To reduce then to the more familiar Focker-Plank form we introduce coefficients

$$A_1 = 4\pi \int p^2 dp N(N+1), \qquad A_2 = 4\pi \int k^2 dk k^2 n(n+1)$$
$$\omega_o^2 = \frac{1}{2}\frac{\int p^2 dp N(N+1) p^2}{\int p^2 dp N(N+1)}, \qquad \Omega_0 = \frac{\int k^2 dk \frac{k^4}{4m} n(n+1)}{\int k^2 dk k^2 n(n+1)}$$
$$B_1 = -\frac{\int p^2 dp N' p^2}{\int p^2 dp N(N+1) p^2}, \qquad B_2 = -\frac{\int k^2 dk k^2 n'}{\int k^2 dk k^2 n(n+1)}$$
$$C_1 = \frac{\int p^2 dp N' p^2}{\int p^2 dp N'}\frac{\int p^2 dp N(N+1)}{\int p^2 dp N(N+1) p^2} - 1, C_2 = \frac{\int k^2 dk \frac{k^4}{4m} n'}{\int k^2 dk k^2 n'}\frac{\int k^2 dk k^2 n(n+1)}{\int k^2 dk \frac{k^4}{4m} n(n+1)} - 1 \quad (29)$$

Note, that in the thermodynamic equilibrium $N' = -N(N+1)/T$, so that $B_1, B_2 = 1/T$ (T is equilibrium temperature) and $C_1, C_2 = 0$. Coefficients $A_{1,2}$ and ω, Ω are complicated even for thermodynamic equilibrium. In the classical limit they give $A_{1,2} = N_{tot}$, n_{tot} - total number of particles, $\omega_o^2 = 3mT$ -average $< k^2 >$ and $\Omega_0 = 5T/4$.

Using notations (29) the collision integrals may be written as

$$I_{\rm coll}(\omega) = \frac{\sigma_0}{2} A_1 \left[\omega^4(\omega^2+\omega_0^2)\left(\frac{\partial n}{\partial\omega} + B_1 n(n+1)\left(1 + \frac{\omega_0^2 C_1}{\omega^2+\omega_0^2}\right)\right)\right]$$
$$I_{\rm coll}(\Omega) = \frac{\sigma_0}{2} A_2 \left[\sqrt{\Omega}(\Omega+\Omega_0)\left(\frac{\partial N}{\partial\Omega} + B_2 N(N+1)\left(1 + \frac{\Omega_0 C_2}{\Omega+\Omega_0}\right)\right)\right] \quad (30)$$

B. Stationary solutions

To find stationary power law solutions to the nonlinear equations (30) we assume that $C_{1,2} \approx 0$ (in this case the resulting solution are sums of power laws (see Eqns. (35-36)). In case of C_1 the dominant contribution to the number density comes from the short wave vector limit of the thermodynamic equilibrium (see Eq. (36)), for which we can assume

$C_1 \approx 0$. The coefficient C_2 on the other hand is generally nonzero, but if the maximal energy of particle n (cutoff of the power law) is much less than the mass m then the term with C_2 is approximately p_{max}/m - much smaller than unity.

Under assumption $C_{1,2} \approx 0$ the Kompaneets-type equations for the occupation numbers n and N become

$$\begin{aligned}\frac{\partial n}{\partial t} &= -\frac{\sigma_0}{2} A_1 \frac{1}{\omega^2} \frac{\partial}{\partial \omega} \left[\omega^4 (\omega^2 + \omega_0^2) \left(\frac{\partial n}{\partial \omega} + B_1 n(n+1) \right) \right] \\ \frac{\partial N}{\partial t} &= -\frac{\sigma_0}{2} A_2 \frac{1}{\sqrt{\Omega}} \frac{\partial}{\partial \Omega} \left[\sqrt{\Omega} (\Omega + \Omega_0) \left(\frac{\partial N}{\partial \Omega} + B_2 N(N+1) \right) \right]\end{aligned} \tag{31}$$

and the the stationary equations $\frac{\partial n}{\partial t} = \frac{\partial N}{\partial t} = 0$ should satisfy

$$\begin{aligned}&\omega^4(\omega^2 + \omega_0^2) \left(n' + B_1 n(n+1) \right) = P_1 \\ &\sqrt{\Omega}(\Omega + \Omega_0) \left(N' + B_2 N(N+1) \right) = P_2\end{aligned} \tag{32}$$

The physical meaning of the constants P_1 and P_2 is the total particle flux in the wave vector space.

Eqns (32) are nonlinear Riccatti-type differential equations. Using a conventional substitution $N'(x) \to U'/U$ they can be reduce to linear equations of the form

$$U'' + U' - G(x)U = 0 \tag{33}$$

It is not generally integrable in quadratures, but in some limits it may have a solution which is a sum of power laws [9]. [1] The best way to search for possible power law solutions is to use the substitution $U(x) \to W \exp\{ \left(\int \sqrt{G(x)} \right)^{-1} dx \}$ which takes a correct account of the leading power law term.

Power law solutions of the equation (32) are possible in the inertial regions - where there is no quantity with a dimension of length: these are regions ω much less or much larger than ω_0, Ω much less or much larger than Ω_0 and N, n much less or much larger than 1.

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[1]For the equilibrium distribution of heavy particle N various approximations for n in the limit $\omega \gg \omega_0$ and $N \ll 1$ are summarized by [10].

In the linear regime $N, n \ll 1$ the stationary solutions are

$$
\begin{aligned}
n &\propto \int \frac{P_1}{\omega^3(\omega^2+\omega_0^2)} d\omega \\
N &\propto \int \frac{P_2\sqrt{\Omega}}{(\Omega+\omega_0)} d\Omega
\end{aligned} \tag{34}
$$

In the strongly nonlinear ($N, n \gg 1$) limit the leading orders in power-laws are

$$
n_\omega \propto \begin{cases} \frac{P_1}{\omega_0\omega^2} + \frac{1}{B\,\omega} & \text{for } \omega \ll \omega_0 \\ \frac{P_1}{\omega^3} + \frac{3}{2B\,\omega} & \text{for } \omega \gg \omega_0 \end{cases} \tag{35}
$$

$$
N_\Omega \propto \begin{cases} \frac{1}{B\Omega} - \frac{P_2}{\Omega_0^{1/2}\Omega^{1/4}} & \text{for } \Omega \ll \Omega_0 \\ \frac{1}{B\Omega} - \frac{P_2}{\Omega^{3/4}} & \text{for } \Omega \gg \Omega_0 \end{cases} \tag{36}
$$

Given the solutions (36), we can also calculate the coefficients Ω_0 and ω_0:

$$
\begin{aligned}
\Omega_0 &= \begin{cases} \frac{k_{max}^2}{8m} & \text{for } n \propto \omega^{-2} \\ \frac{k_{max}^2}{12m} & \text{for } n \propto \omega^{-3} \end{cases} \\
\omega_0 &= p_{max}/\sqrt{3}
\end{aligned} \tag{37}
$$

where k_{max} and p_{max} are the upper cut-offs limits for the corresponding power laws.

We can verify that the integral in the expressions for the coefficients (Eq. (29) converge at $k, p = 0$. Note also, that for massive particles N, the leading term in the small wave vector limit corresponds to the low frequency asymptotics of the thermodynamic equilibrium spectrum.

Equations (35-37) give the selfconsistent Kolmogorov-type solutions to turbulent interaction of two types of waves. They are qualitatively shown in Fig. (2). In all cases the particles are flowing to small k, p with a constant flux in phase space. If the sources of particles are located at $k, p \gg m$ (Fig. 2 (a)), so that $\omega_0 \approx m$, then the low energy tail of the distribution of massless particles ($\omega \ll \omega_0$) is $n \propto \omega^{-2}$. Alternatively, if the sources of particles are located at $k, p \ll m$ (Fig. 2 (b)), then the low energy tail of the distribution of massless particles is $n \propto \omega^{-3}$.

V. CONCLUSION

We have considered turbulent 4-wave interaction of two types of waves with different dispersion laws: acoustic and electromagnetic-type. Both types of waves were assumed to obey a quantum boson statistics. We found possible stationary Kolmogorov-type spectra in the inertial ranges k, $p \gg m$ and k, $p \ll m$. To reach the stationary solutions the particles should be generated at finite momenta k and p and flow to small wave numbers. In the "ultrarelativistic limit" k, $p \gg m$, the turbulent interaction of two types of waves resembles self interaction of semiacoustic waves with the dispersive correction ensuring the applicability of the weak turbulence limit. In the small wave vector limit k, $p \ll m$ the collisional integral may be reduced to diffusion-type differential equation. In this limit, the small wave number asymptotic of massive particles is dominated by the thermal equilibrium tail $N \propto \Omega^{-1}$, while the small wave number asymptotic of massless particles is steeper: $n \propto \omega^{-2} \div \omega^{-3}$. Arguably, the massless particles n will eventually reach $k = 0$ to be absorbed in the Bose condensate.[2]

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[1] Zakharov V.E., L'vov V.S., Falkovich G., *Kolmogorov spectra of turbulence*, 1992, Berlin ; New York : Springer-Verlag

[2] Balk A, 1994, Phys Lett A 187, 302

[3] Newell A.C. & Aucoin P.J., J. Fluid Mech., 1971, 49, 593

[4] Zakharov V.E. & Sagdeev R.Z. 1970, Sov. Phys. Doklady, 15, 439

[5] Lvov V.S., Lvov Yu, Newel A.C. & Zakharov V.E. 1997, Phys. Rev. E, 56, 390

[2] Limitations of the kinetic approach to Bose condensation were discussed by Kogan et al. (1992)

[6] Kompaneets A.S. 1956, Sov, Phys. JETP, 31, 876

[7] Landau L. & Lifshits 1977, *Quantum Mechanics*, Oxford ; New York : Pergamon Press

[8] Zakharov V.E. 1983, *Basic plasma physics*, Galeev A.A. & Sudan R.N., eds, North-Holland Pub. ; New York, N.Y.

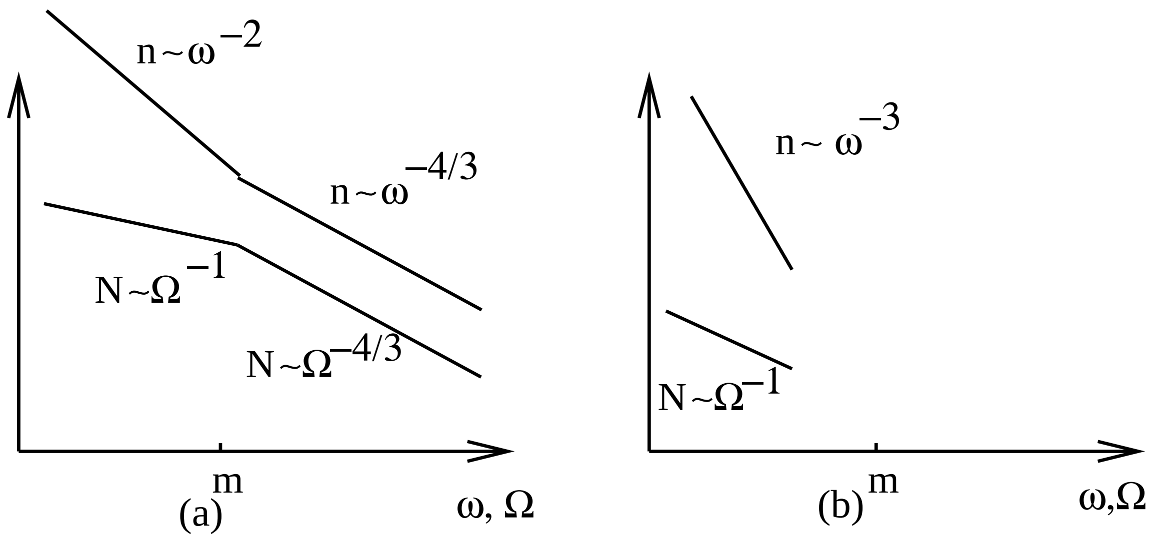
[9] Kamke E. 1969, *Differentialgleichungen*, Leipzig : Akademische Verlagsgesellschaf

[10] Sunyaev R.A.& Titarchuk L.G. 1980, Astron. Astroph. 86, 121

[11] Kagan Yu. M., Svistunov B.V., Shlyapnikov G. V. 1992, Sov. Phys. - JETP, 74, 279

FIGURE CAPTIONS

FIG. I. Stationary power law solutions of the system (12). (a) source of particles is located at $k, p \gg m$, (b) source of particles is located at $k, p \ll m$



n ~ ω−2
n ~ ω−4/3
N ~ Ω−1
N ~ Ω−4/3
m
(a)
ω, Ω
n ~ ω−3
N ~ Ω−1
(b)
m
ω,Ω